MAT 219 (lecture 5)

Review of Chapter 2
Begin Chapter 7. → §7.1: Introduction (Phase Planes) / §7.2: Review of matrices

Next: Independence, Eigenvalues, Eigenvectors

Review of Chapter 2: First Order DE

Separable: $M(x)\,dx = N(y)\,dy$

Integrate: $\int M(x)\,dx = \int N(y)\,dy$

Homogeneous: $y' = \frac{P(x,y)}{Q(x,y)}$

P & Q polynomials
all $x^n y^m$ have total power $(n+m)$ same

Substitute: $x \mapsto 1$, $y \mapsto v$, $y' \mapsto v + xv'$

(Eqn becomes separable)

Linear: $y' + p(x)y = g(x)$

Integrating Factor: $\mu = e^{\int p(x)\,dx}$

Mult. & Int.: $\mu y = \int \mu g\,dx$

Check: $(\mu y)' = \mu y' + \mu p y$

Exact: $M(x,y)\,dx + N(x,y)\,dy = 0$ where $\partial/\partial y\, M = \partial/\partial x\, N$

Integrate for potential function (least common sum)

$$f(x,y) = \text{lcs}\begin{cases}\int M(x,y)\,dx \\ \int N(x,y)\,dy\end{cases}$$

Solution $f(x,y) = c$

Integrating Factors for Exact ??

$\mu = e^{\int \frac{M_y - N_x}{N} dx}$ or $\mu = e^{\int \frac{N_x - M_y}{M} dy}$

$\int \frac{M_y - N_x}{N} \frac{dx\,\cancel{dy}}{\cancel{dy}}$ $\qquad$ $\int \frac{N_x - M_y}{M} \frac{dy\,\cancel{dx}}{\cancel{dx}}$

Existence & Uniqueness Thms

Linear: $y' + p(x)y = g(x)$ with $y(x_0) = y_0$

- ~~There is one and only one solution as long as p & g are continuous at x_0~~
- ~~Solution is defined on interval containing x_0 where p & g are cont.~~

→ Vertical solutions at x_0 where $p(x)$ or $g(x)$ undef.

General IVP: $y' = f(x,y)$ with $y(x_0) = y_0$

- ~~There is solution if f cont. at (x_0, y_0)~~
- ~~Only one solution if $\partial/\partial f$ cont at (x_0, y_0)~~

→ Vertical solutions at x_0 where $f(x,y)$ undef.

→ Turning points $(\pm)$ at y_0 where $f(x,y)$ undef.

→ Extra solutions at y_0 where $\partial/\partial y\, f(x,y)$ undef.

Begin Chapter 7.

Systems of Linear Differential Equations

Systems of DE appear when multiple things change over time & interact.

EX Predator – Prey

x = # rabbits
y = # wolves
(functions of time t)

$$\frac{dx}{dt} = x(\underbrace{r_1}_{\text{birth rate}} - \underbrace{\alpha y}_{\text{death rate by wolves}})$$

$$\frac{dy}{dt} = y(\underbrace{\beta x}_{\text{birth rate}} - \underbrace{r_2}_{\text{death rate}})$$

How to graph results:
<u>phase plane</u> x vs. y (ignore t)

EX Competing Species

x = # wolves
y = # lions
(functions of time t)

$$\frac{dx}{dt} = x(r_1 - \alpha y)$$
$$\frac{dy}{dt} = y(r_2 - \beta x)$$

EX Logistic Competition

x = # rabbits
y = # sheep

$$\frac{dx}{dt} = r_1 x\left(1 - \frac{x+y}{M_1}\right)$$
$$\frac{dy}{dt} = r_2 y\left(1 - \frac{x+y}{M_2}\right)$$

EX Directed Fire

x = # blues
y = # reds

$$\frac{dx}{dt} = -\underbrace{\alpha}_{P(\text{red hits blue he aims at})} y$$
$$\frac{dy}{dt} = -\beta x$$

(Note: Combat parity $\frac{x^2}{y^2} = \frac{\alpha}{\beta}$

→ 2x troops ⟷ 4x training
see Rambo.)

EX Undirected Fire

x = # blues
y = # reds

$$\frac{dx}{dt} = -\alpha x y$$
$$\frac{dy}{dt} = -\beta x y$$

(parity: $\frac{x}{y} = \frac{\alpha}{\beta}$)

EX Mixed Fire

x = # blues
y = # reds

$$\frac{dx}{dt} = -\alpha x y$$
$$\frac{dy}{dt} = -\beta x$$

(Blues hidden
Reds are visible)

(parity: $\frac{x}{y^2} = \frac{\alpha}{2\beta}$)

→ Simplest systems: <u>Linear</u>

$$\frac{dx}{dt} = p_1(t)x + q_1(t)y + g_1(t)$$
$$\frac{dy}{dt} = p_2(t)x + q_2(t)y + g_2(t)$$

<u>EX</u>

$$x' = 2t^2 x - 3ty + 1$$
$$y' = -tx + 4t^3 y - 12t$$

<u>Note</u>: Sometimes we will want >2 equations... maybe x & y is bad notation.

$$\frac{dx_1}{dt} = p_{11} x_1 + p_{12} x_2 + g_1$$
$$\frac{dx_2}{dt} = p_{21} x_1 + p_{22} x_2 + g_2$$

A <u>solution</u> to a system of equations is functions for x_1 & x_2 etc.

<u>EX</u>

$$x_1' = 3x_1 - x_2$$
$$x_2' = 4x_1 - 2x_2$$

has a solution
$$x_1 = e^{2t} + e^{-t}$$
$$x_2 = e^{2t} + 4e^{-t}$$

General solution has undetermined constants

$$x_1 = c_1 e^{2t} + c_2 e^{-t}$$
$$x_2 = c_1 e^{2t} + 4c_2 e^{-t}$$

Initial value problems must ~~specify~~ give start values for x_1 & x_2

$$x_1' = 3x_1 - x_2 \qquad \text{with} \qquad x_1(0) = 3$$
$$x_2' = 4x_1 - 2x_2 \qquad\qquad\qquad x_2(0) = 6$$

⇝ solve for c_1 & c_2

$$3 = x_1(0) = c_1 \cancel{e^0}\,1 + c_2 \cancel{e^0}\,1$$
$$-\quad 6 = x_2(0) = c_1 \cancel{e^0}\,1 + 4c_2 \cancel{e^0}\,1$$

$$-3 = -3c_2 \Rightarrow c_2 = 1$$
$$\Rightarrow c_1 = 2$$

<u>Solution</u>:
$$\boxed{\begin{aligned} x_1 &= 2e^{2t} + e^{-t} \\ x_2 &= 2e^{2t} + 4e^{-t} \end{aligned}}$$